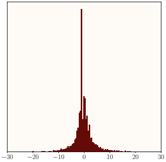
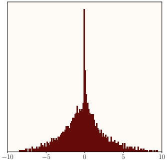
Spectra of sparse random graphs

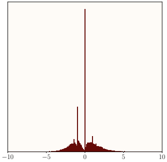
Simon Coste — LMV, 2021



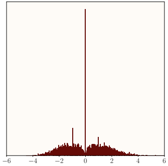
Bible



Emails



Pgp



Powergrid

I

FACTS, PICTURES and QUESTIONS

Examples of random graphs models

Erdős-Rényi graphs:

- $V = \{1, \ldots, n\}$.
- Put each potential edge (u, v) in E independently with probability p.

Random trees:

- $\mathcal{T}_n$ = set of trees on n vertices. $|\mathcal{T}_n| = n^{n-2}$ (Cayley's formula)
- Take G uniformly at random in $\mathcal{T}_n$.

Random regular graphs

- $\mathcal{G}_{n,d}$ = set of d-regular graphs with n vertices.
- Take G uniformly at random in $\mathcal{G}_{n,d}$.

Pictures: eigenvalues of uniform 3-regular graphs

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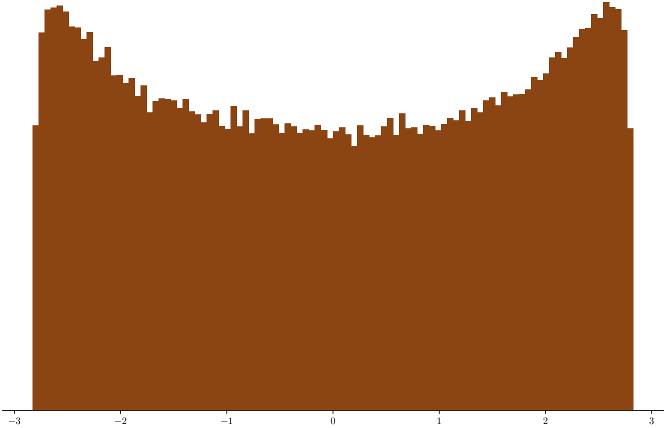
Histogram of eigenvalues of a uniform 3-regular graph on $n = 10000$ vertices



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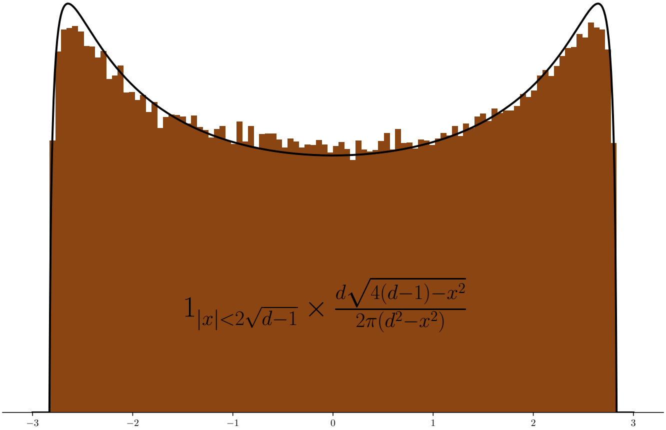

Limiting distribution = Kesten-McKay distribution

Absolutely continuous, bounded support, bounded density

Pictures: eigenvalues of Erdős-Rényi graphs, DENSE case

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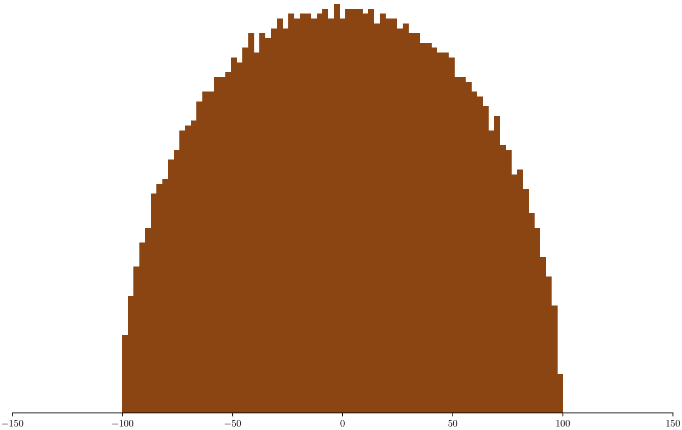
Histogram of eigenvalues of an Erdős-Rényi graph
$n = 10000$ vertices, $p = 1/2$; the average degree is $n/2$.



Pictures: eigenvalues of Erdős-Rényi graphs, DENSE case

Histogram of eigenvalues of an Erdős-Rényi graph
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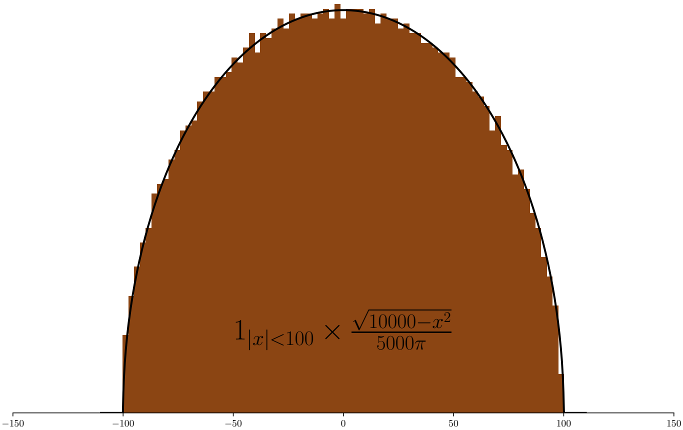

This is Wigner's semicircle distribution (rescaled).
Closed form, absolutely continuous, bounded support, bounded density.

Pictures: eigenvalues of Erdős-Rényi graphs, SPARSE case

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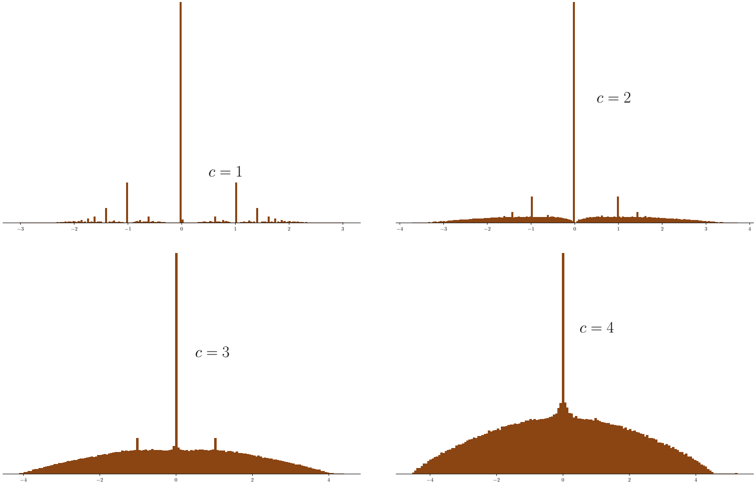
Histogram of eigenvalues of an Erdős-Rényi graph
$n = 10000$ vertices, $p = c/n$; the average degree is c.



Pictures: eigenvalues of uniform trees

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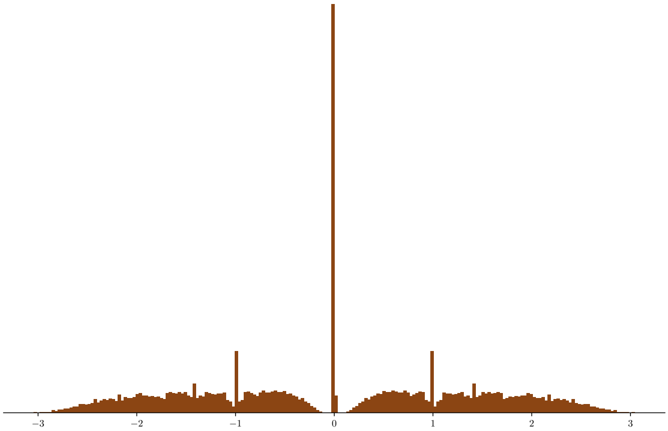
Histogram of eigenvalues of a uniform tree on $n = 10000$ vertices (averaged over 100 realizations).



II

BENJAMINI-SCHRAMM CONVERGENCE

Definition of BS convergence

$\mathcal{G}_*$ = set of rooted graphs (g,o) with a countable set of vertices
$(g,o)_t$= graph induced the ball of radius t around the root

Similarity between rooted graphs:
$\mathrm{Sim}((g,o),(g',o')) = \max\{t \in \mathbb{N} : (g,o)_t \text{ and } (g',o')_t \text{ are isomorphic}\}$

Local distance on $\mathcal{G}_*$:

$$d((g,o),(g',o')) = (\mathrm{Sim}((g,o),(g',o'))+1)^{-1} \tag{1}$$

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Definition. Let G_n be a sequence of finite graphs.

- We root them uniformly at random: $o_n \sim \mathrm{Uniform}(V_n)$ and take the connected component of the root, noted $G_n(o_n)$.
- (G_n,o_n) is now a random rooted (finite) graph.
- We say that G_n **converges in the Benjamini-Schramm sense** towards some random rooted graph (G,o) if the distribution of (Gn,o_n) converges weakly to the distribution of (G,o).

Some examples of local weak convergence

Random graph model	**Benjamini-Schramm limit**
Erdős-Rényi $(n, c/n)$	Galton-Watson Poisson(c)
Uniform trees	Skeleton tree
Random d-regular graphs	d-regular tree
Preferential attachment	Polya point-tree

Eigenvalues of graphs

G = finite graph with n vertices, with adjacency matrix A

$\lambda_1, \ldots, \lambda_n$ = eigenvalues of A

$\mu_{G_n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}$ Empirical Spectral Distribution

Spectral continuity

Kolmogorov-Smirnov continuity [Abért, Thom, Virag, 2015]

Suppose that

$$G_n \xrightarrow[n\to\infty]{\text{(BS)}} (G,o)$$

where (G,o) is a random rooted graph with distribution ρ.

Spectral continuity

Kolmogorov-Smirnov continuity [Abért, Thom, Virag, 2015]

Suppose that

$$G_n \xrightarrow[n\to\infty]{(\mathrm{BS})} (G,o)$$

where (G,o) is a random rooted graph with distribution ρ.

Then there is a probability distribution μ_ρ such that

$$\sup_{t\in\mathbb{R}} |F_{\mu_{G_n}}(t) - F_{\mu_\rho}(t)| \to 0 \quad (2)$$

where F is the cumulative distribution function.

Representation of the limiting distribution

(G,o) = rooted graph and let A be its adjacency operator on $\ell^2(V)$.
$(e_v : v \in V)$ = canonical basis of $\ell^2(V)$

Herglotz theory

There is a probability measure $\mu_{(G,o)}$ such that for any $z \in \mathbb{C}_+$

$$\langle e_o, (A-z)^{-1} e_o \rangle = \int_{\mathbb{R}} \frac{1}{\lambda - z} \mu_{(G,o)}(\mathrm{d}\lambda). \tag{3}$$

Representation of the limiting distribution

Suppose that $G_n \xrightarrow{(BS)} (G,o)$ with distribution ρ. Then $\mu_{G_n} \to \mu_\rho$ and

$$\mu_\rho = \mathbf{E}_\rho[\mu_{(G,o)}]. \tag{4}$$

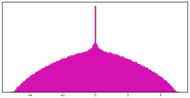
μ_c = notation for μ_ρ with $\rho = \mathrm{GaltonWatson}(\mathrm{Poisson}(c))$.

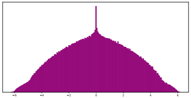
III

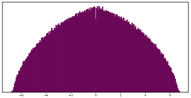
SOME RESULTS

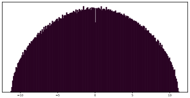
Convergence towards semi-circle

Histograms of eigenvalues of Erdős-Rényi graphs with parameter c/n and size $n = 5000$ (average over 100 realizations):


(a) $c = 5$


(b) $c = 8$


(c) $c = 10$


(d) $c = 30$

Convergence towards semi-circle

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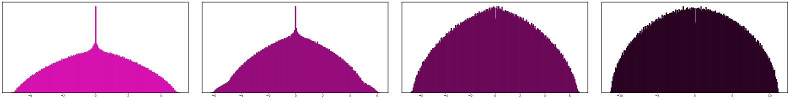


(e) $c = 5$ (f) $c = 8$ (g) $c = 10$ (h) $c = 30$

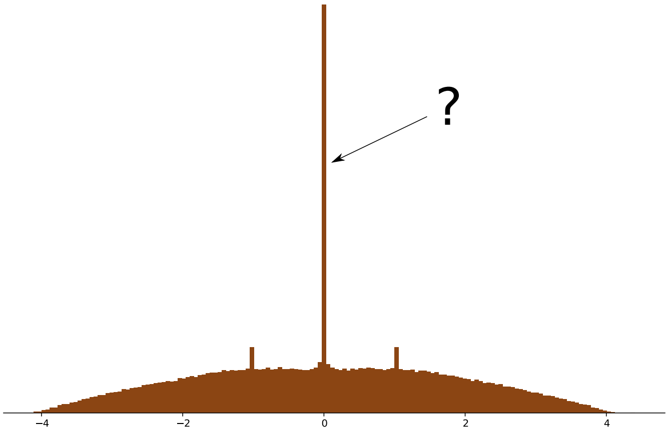
[Jung, Lee, 2017]

$$\frac{\mu_c}{\sqrt{c}} \xrightarrow[c \to \infty]{(d)} \text{Wigner semicircle distribution}$$

Atom at zero

Atom at zero

'Histogram' of μ_c with $c = 3$



Atom at zero: computation is feasible

[Bordenave, Lelarge, Salez 2015]: atom at zero for Poisson(c) GW trees

$$\mu_c(\{0\}) = \mathrm{e}^{-c\alpha} + c\alpha\mathrm{e}^{-c\alpha} + \alpha - 1$$

where α is the smallest solution of $x = \mathrm{e}^{-c\mathrm{e}^{-cx}}$ in $(0,1)$.
+ generalization to any unimodular GW trees

[Bauer, Golinelli, 2000]: atom at zero for the skeleton tree

$$\mu_{\mathrm{skel}}(\{0\}) = 2\beta - 1$$

where $\beta \approx 0.567\ldots$ is the unique solution in $(0,1)$ of $x = \mathrm{e}^{-x}$.

Existence of a continuous part

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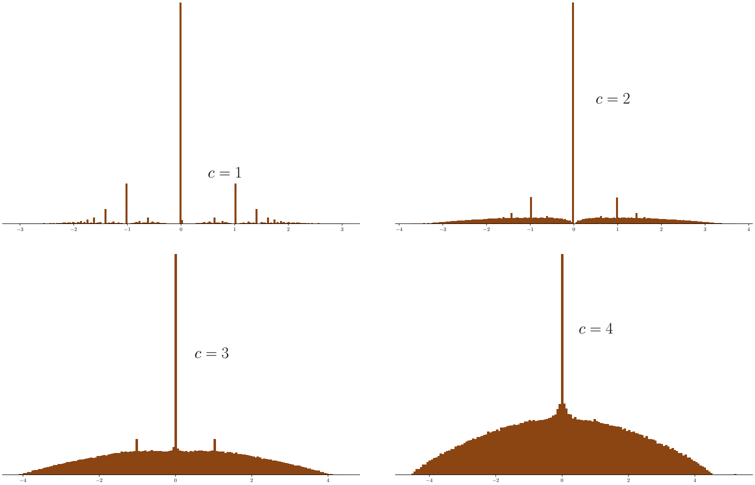


Existence of a continuous part

[Bordenave, Sen, Virag, 2015]

μ_c has a continuous part

$$\Leftrightarrow$$

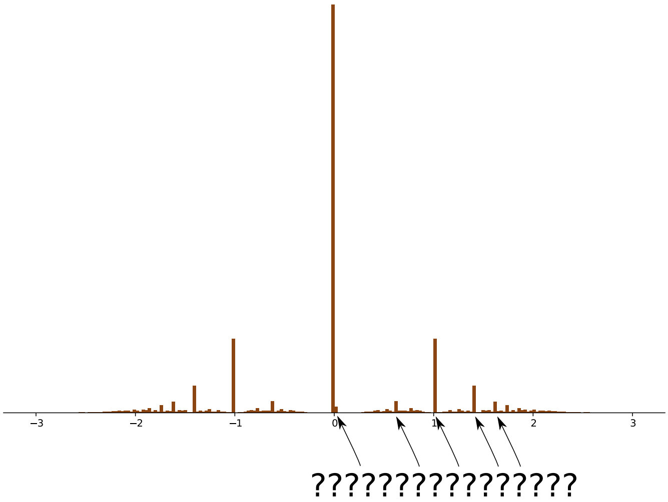
$$c > 1$$

[Arras, Bordenave, 2021]

For any ε there is a c_ε such that if $c > c_\varepsilon$, then the **absolutely continuous** part of μ_c has mass $> 1 - \varepsilon$.

Atoms of **unimodular** trees: **where** are they?

'Histogram' of μ_c with $c = 1$



Atoms of **unimodular** trees

[Salez 2016]

T= some unimodular random tree with distribution ρ
$\mu_\rho = \mathbf{E}[\mu_{(T,o)}]$

$$\{\text{atoms of } \mu_\rho\} \subset \{\text{totally real algebraic integers }\} := \mathbb{A}$$

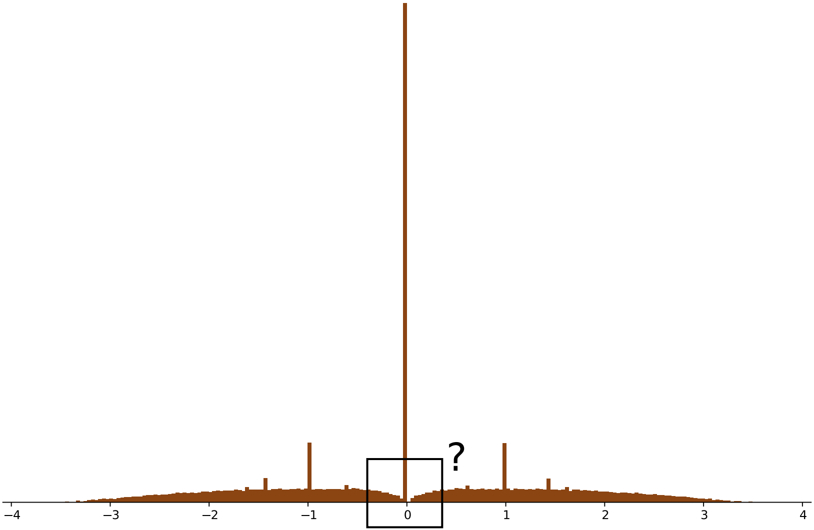
$\mathbb{A}$ = roots of polynomial P with integer coefficients, with only real roots.
$\mathbb{A}$ is dense in $\mathbb{R}$.

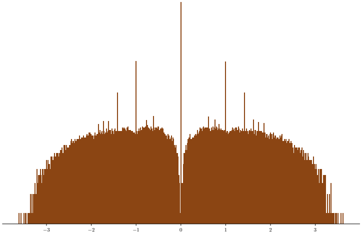
Consequences:

- atoms of μ_c = $\mathbb{A}$
- atoms of μ_{skel} = $\mathbb{A}$

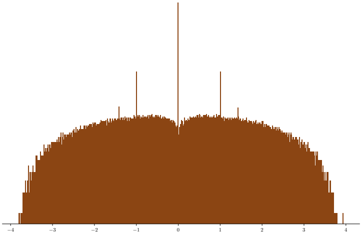
[Bencs, Mészáros, 2020] : generalization to matching measures of graphs.

What happens **around** zero ?

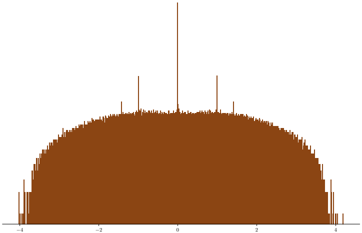




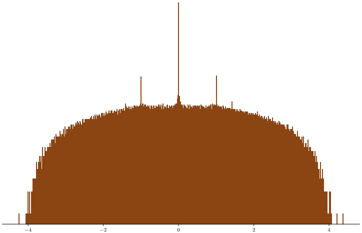
$c = 2$



$c = 2,6$



$c = 2,8$



$c = 3$

What happens at zero ?

Definition: we say that a measure μ has extended states at E if

$$\lim_{\varepsilon \to 0} \frac{\mu((E-\varepsilon, E+\varepsilon)) - \mu(\{E\})}{2\varepsilon} > 0$$

[C, Salez, 2018]

μ_c has extended states at zero

$$\Leftrightarrow$$

$$c > \mathrm{e}$$

+ easy generalization [C, 19+]: μ_{skel} has no extended states at zero.

I don't have answers to these questions

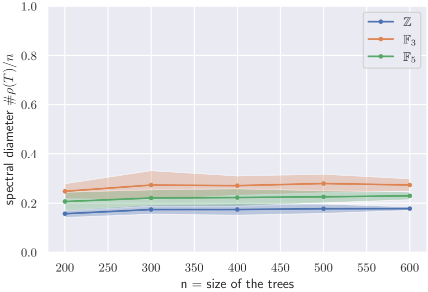
- Does μ_{skel} have a continuous part ?
- What is the nature of the continuous part of μ_c?
- Is there a unimodular tree with singular continuous part?
- Is there a unimodular tree with only one semi-infinite path and a continuous part?
- What is the value of every atom of μ_c?
- What about the support of these measures, or the support of their continuous parts?
- Can you translate some Anderson localization results in this setup?

A reachable (?) conjecture

Trees have a linear spectral diameter

Let T_n be a uniform tree on n vertices.
There is a constant c such that whp the number of distinct eigenvalues of T_n is $\geqslant cn$.

If true, then μ_{skel} has a continuous part (Justin Salez).



Merci !

(le plus vieil arbre de Versailles, planté en 1668)