Spectra of sparse random graphs

Simon Coste — LMV, 2021
FACTS, PICTURES and QUESTIONS
Examples of random graphs models

Erdős-Rényi graphs:

- $V = \{1, \ldots, n\}$.
- Put each potential edge $(u, v)$ in $E$ independently with probability $p$.

Random trees:

- $\mathcal{T}_n =$ set of trees on $n$ vertices. $|\mathcal{T}_n| = n^{n-2}$ (Cayley’s formula)
- Take $G$ uniformly at random in $\mathcal{T}_n$.

Random regular graphs

- $\mathcal{G}_{n,d} =$ set of $d$-regular graphs with $n$ vertices.
- Take $G$ uniformly at random in $\mathcal{G}_{n,d}$.
Histogram of eigenvalues of a uniform 3-regular graph on \( n = 10000 \) vertices

Limiting distribution = Kesten-McKay distribution

Absolutely continuous, bounded support, bounded density
Histogram of eigenvalues of a uniform 3-regular graph on $n = 10000$ vertices

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Limiting distribution = Kesten-McKay distribution
Absolutely continuous, bounded support, bounded density

\[ \frac{1}{|x| < 2\sqrt{d-1}} \times \frac{d\sqrt{4(d-1) - x^2}}{2\pi(d^2 - x^2)} \]
Histogram of eigenvalues of an Erdős-Rényi graph

$n = 10000$ vertices, $p = 1/2$; the average degree is $n/2$.

This is Wigner's semicircle distribution (rescaled).

Closed form, absolutely continuous, bounded support, bounded density.
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Pictures: eigenvalues of Erdős-Rényi graphs, DENSE case

Histogram of eigenvalues of an Erdős-Rényi graph

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This is Wigner’s semicircle distribution (rescaled).
Closed form, absolutely continuous, bounded support, bounded density.
Pictures: eigenvalues of Erdős-Rényi graphs, SPARSE case

$n = 10000$ vertices, $p = c/n$; the average degree is $c$. 

$c = 1$

$c = 2$

$c = 3$

$c = 4$
Histogram of eigenvalues of an Erdős-Rényi graph

$n = 10000$ vertices, $p = c/n$; the average degree is $c$. 

$c = 1$

$c = 2$

$c = 3$

$c = 4$
Pictures: eigenvalues of uniform trees
Histogram of eigenvalues of a uniform tree on $n = 10000$ vertices (averaged over 100 realizations).
II

BENJAMINI-SCHRAMM CONVERGENCE
Definition of BS convergence

\[ G_* = \text{set of rooted graphs } (g,o) \text{ with a countable set of vertices} \]
\[ (g,o)_t = \text{graph induced the ball of radius } t \text{ around the root} \]

Similarity between rooted graphs:
\[ \text{Sim}((g,o),(g',o')) = \max\{t \in \mathbb{N} : (g,o)_t \text{ and } (g',o')_t \text{ are isomorphic}\} \]

Local distance on \( G_* \):
\[ d((g,o),(g',o')) = (\text{Sim}((g,o),(g',o')) + 1)^{-1} \quad (1) \]
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(1)

**Definition.** Let \( G_n \) be a sequence of finite graphs.

- We root them uniformly at random: \( o_n \sim \text{Uniform}(V_n) \) and take the connected component of the root, noted \( G_n(o_n) \).
- \( (G_n, o_n) \) is now a random rooted (finite) graph.
- We say that \( G_n \) **converges in the Benjamini-Schramm sense** towards some random rooted graph \( (G, o) \) if the distribution of \( (G_n, o_n) \) converges weakly to the distribution of \( (G, o) \).
## Some examples of local weak convergence

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<th>Benjamini-Schramm limit</th>
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<td>Erdős-Rényi ((n, c/n))</td>
<td>Galton-Watson Poisson((c))</td>
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<td>Uniform trees</td>
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<td>Random (d)-regular graphs</td>
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<td>Preferential attachment</td>
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Eigenvalues of graphs

\( G \) = finite graph with \( n \) vertices, with adjacency matrix \( A \)

\( \lambda_1, \ldots, \lambda_n = \text{eigenvalues of } A \)

\[ \mu_{G_n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i} \]  

Empirical Spectral Distribution
Suppose that

\[ G_n \xrightarrow{(BS)} (G, o) \]

where \((G, o)\) is a random rooted graph with distribution \(\rho\).
Spectral continuity

Kolmogorov-Smirnov continuity [Abért, Thom, Virag, 2015]

Suppose that

\[ G_n \xrightarrow{(BS)\;n \to \infty} (G,o) \]

where \((G,o)\) is a random rooted graph with distribution \(\rho\).

Then there is a probability distribution \(\mu_{\rho}\) such that

\[
\sup_{t \in \mathbb{R}} |F_{\mu_{G_n}}(t) - F_{\mu_{\rho}}(t)| \to 0
\]

where \(F\) is the cumulative distribution function.
\((G, o) = \text{rooted graph and let } A \text{ be its adjacency operator on } \ell^2(V).\n\]
\[(e_v : v \in V) = \text{canonical basis of } \ell^2(V)\n\]

**Herglotz theory**

There is a probability measure \(\mu_{(G,o)}\) such that for any \(z \in \mathbb{C}_+\)

\[
\langle e_o, (A - z)^{-1}e_o \rangle = \int_{\mathbb{R}} \frac{1}{\lambda - z} \mu_{(G,o)}(d\lambda).
\]

(3)

**Representation of the limiting distribution**

Suppose that \(G_n \xrightarrow{(BS)} (G, o)\) with distribution \(\rho\). Then \(\mu_{G_n} \rightarrow \mu_\rho\) and

\[
\mu_\rho = \mathbb{E}_\rho[\mu_{(G,o)}].
\]

(4)

\(\mu_c = \text{notation for } \mu_\rho \text{ with } \rho = \text{GaltonWatson}(\text{Poisson}(c)).\)
III

SOME RESULTS
Convergence towards semi-circle

Histograms of eigenvalues of Erdős-Rényi graphs with parameter $c/n$ and size $n = 5000$ (average over 100 realizations):

(a) $c = 5$
(b) $c = 8$
(c) $c = 10$
(d) $c = 30$

$\mu_c \sqrt{c} (d) \xrightarrow{c \to \infty} Wigner$ semicircle distribution
Convergence towards semi-circle

Histograms of eigenvalues of Erdős-Rényi graphs with parameter $c/n$ and size $n = 5000$ (average over 100 realizations):

(e) $c = 5$
(f) $c = 8$
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(h) $c = 30$

[Jung, Lee, 2017]

$$\frac{\mu_c}{\sqrt{c}} \xrightarrow{(d)} Wigner \text{ semicircle distribution}$$
Atom at zero
‘Histogram’ of $\mu_c$ with $c = 3$
Atom at zero: computation is feasible

[Bordenave, Lelarge, Salez 2015]: atom at zero for Poisson(c) GW trees

\[ \mu_c(\{0\}) = e^{-c\alpha} + c\alpha e^{-c\alpha} + \alpha - 1 \]

where \( \alpha \) is the smallest solution of \( x = e^{-ce^{-cx}} \) in \((0, 1)\).

+ generalization to any unimodular GW trees

[Bauer, Golinelli, 2000]: atom at zero for the skeleton tree

\[ \mu_{skel}(\{0\}) = 2\beta - 1 \]

where \( \beta \approx 0.567 \ldots \) is the unique solution in \((0, 1)\) of \( x = e^{-x} \).
Existence of a continuous part
Existence of a continuous part

- $c = 1$
- $c = 2$
- $c = 3$
- $c = 4$
<table>
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<td><strong>[Bordenave, Sen, Virag, 2015]</strong></td>
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<tr>
<td>$\mu_c$ has a continuous part</td>
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<tr>
<td>$\Leftrightarrow$</td>
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<td>$c &gt; 1$</td>
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<tr>
<td><strong>[Arras, Bordenave, 2021]</strong></td>
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<td>For any $\epsilon$ there is a $c_\epsilon$ such that if $c &gt; c_\epsilon$, then the <strong>absolutely continuous</strong> part of $\mu_c$ has mass $&gt; 1 - \epsilon$.</td>
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Atoms of unimodular trees: where are they?

‘Histogram’ of $\mu_c$ with $c = 1$
Atoms of **unimodular** trees

[Salez 2016]

\[ T = \text{some unimodular random tree with distribution } \rho \]
\[ \mu_\rho = \mathbf{E}[\mu_{(T,o)}] \]

\{\text{atoms of } \mu_\rho\} \subset \{\text{totally real algebraic integers }\} := \mathbb{A}

\[ \mathbb{A} = \text{roots of polynomial } P \text{ with integer coefficients, with only real roots.} \]
\[ \mathbb{A} \text{ is dense in } \mathbb{R}. \]

Consequences:

\[ \Rightarrow \text{ atoms of } \mu_c = \mathbb{A} \]
\[ \Rightarrow \text{ atoms of } \mu_{\text{ske}} = \mathbb{A} \]

What happens **around** zero?
Extra simulations in log scale

\[ c = 2 \]

\[ c = 2, 6 \]

\[ c = 2, 8 \]

\[ c = 3 \]
**Definition:** we say that a measure $\mu$ has extended states at $E$ if

$$\lim_{\varepsilon \to 0} \frac{\mu((E-\varepsilon, E+\varepsilon)) - \mu(\{E\})}{2\varepsilon} > 0$$

[C, Salez, 2018]

$\mu_c$ has extended states at zero

$\iff$

$c > e$

+ easy generalization [C, 19+]: $\mu_{\text{skep}}$ has no extended states at zero.
I don’t have answers to these questions

★ Does $\mu_{\text{skel}}$ have a continuous part?
★ What is the nature of the continuous part of $\mu_c$?
★ Is there a unimodular tree with singular continuous part?
★ Is there a unimodular tree with only one semi-infinite path and a continuous part?
★ What is the value of every atom of $\mu_c$?
★ What about the support of these measures, or the support of their continuous parts?
★ Can you translate some Anderson localization results in this setup?
A reachable (?) conjecture

Trees have a linear spectral diameter

Let $T_n$ be a uniform tree on $n$ vertices. There is a constant $c$ such that whp the number of distinct eigenvalues of $T_n$ is $\geq cn$.

If true, then $\mu_{\text{ske1}}$ has a continuous part (Justin Salez).
Merci !

(le plus vieil arbre de Versailles, planté en 1668)